UDC 519.832.4

SUPEROPTIMAL MIXED STRATEGIES IN ANTAGONISTIC GAMES AS THE ADVANTAGED SUBSET OF THE OPTIMAL MIXED STRATEGIES SET

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There has been defined the set of the most advantaged optimal mixed strategies, named the superoptimal mixed strategies, for applying them and obtaining the potential profit in the relevant antagonistic games. The stated principle of the superoptimality is based on the Bayes-Laplace criterion.
Keywords: antagonistic game, optimal mixed strategy, Bayes-Laplace criterion.

Problem description. There are the widespread conflict systems, which are modeled with the antagonistic games theory. When the antagonistic game is solved in pure strategies, then it is simple for both players to make the optimal and logically founded decision. When the solution is in the mixed strategies, then for making the optimal decision every player should practice its optimal mixed strategy [1, 2]. Though the known optimality principle gives the set of optimal game solutions, some of those optimal solutions may have an advantage above the rest [3, 4]. Questions of selecting one of those optimal solutions were discussed in [3, 5]. The head principle of discriminating the solutions, satisfying the optimality principle, is in aggregating the post-affects of their application [6, 7]. A post-affect is an actual value of the game issue in some situation [6]. However, the known aggregation of the post-affects of optimal solutions application refers to only finite number of pure strategies of the player, and it is just the simple summation of actual values of the game issue [7].

Setting the paper task. Going into particulars, the paper [4] were devoted to the definition of suchlike advantaged optimal solutions, but only in pure strategies. They had been divided into two sets – the set of nonstrictly rational pure strategies $\tilde{S}_r$ and the set of strictly rational pure strategies S_r, where $S_r \subseteq \tilde{S}_r$ and the set $\tilde{S}_r$ is a subset of the optimal pure strategies set S_{opt}. And there had been shown, what profit the player gains, if it uses the set of the strictly or the nonstrictly rational strategies by the other player receding from its set of optimal pure strategies. This profit on average is greater than the game value V_{opt} for the first player, and lesser than the game value V_{opt} for the second. The current paper will resolve the problem of defining the most advantaged optimal pure strategies from the set $\tilde{S}_r$ by $\tilde{S}_r \neq \varnothing$ and $\left|S_{\text{opt}}\right| > 1$, and also will define the most advantaged optimal mixed strategies from the set of all the optimal mixed strategies. The last declared definition will generalize the concept of applying the advantaged optimal solutions from the set of all the optimal solutions.

Defining the advantaged subset of the optimal mixed strategies set. May a surface $K(x, y)$ be the kernel of an antagonistic game, where $x \in X$ is a pure strategy of the first player, $y \in Y$ is a pure strategy of the second player, and this surface is defined on the Cartesian product $X \times Y$ of the pure strategies sets of those players. If $X_{\text{opt}} \subset X$ is the nonempty set of the first player optimal pure strategies, and $Y_{\text{opt}} \subset Y$ is the nonempty set of the second player optimal pure strategies, then there are four definitions, stated in the paper [4].

Definition 1. In the antagonistic game with the kernel $K(x, y)$ an optimal pure strategy $x_r \in X_r \subset X_{\text{opt}}$ of the first player is called the strictly rational pure strategy, if $\forall\ x_0 \in X_{\text{opt}} \setminus X_r$, $\forall\ y \notin Y_{\text{opt}}$ and $\forall\ x_r \in X_r$ there is the identity $V_{\text{opt}} = K(x_0, y)$ and the inequality $V_{\text{opt}} < K(x_r, y)$, where $X_r \subseteq \tilde{X}_r \subseteq X_{\text{opt}}$ is the set of all the strictly rational pure strategies of the first player.

Definition 2. In the antagonistic game with the kernel $K(x, y)$ an optimal pure strategy $y_r \in Y_r \subset Y_{\text{opt}}$ of the second player is called the strictly rational pure strategy, if $\forall\ y_0 \in Y_{\text{opt}} \setminus Y_r$, $\forall\ x \notin X_{\text{opt}}$ and $\forall\ y_r \in Y_r$ there is the identity $K(x, y_0) = V_{\text{opt}}$ and the inequality $K(x, y_r) < V_{\text{opt}}$, where $Y_r \subseteq \tilde{Y}_r \subseteq Y_{\text{opt}}$ is the set of all the strictly rational pure strategies of the second player.

Definition 3. In the antagonistic game with the kernel $K(x, y)$ an optimal pure strategy $\tilde{x}_r \in \tilde{X}_r \subset X_{\text{opt}}$ of the first player is called the nonstrictly rational pure strategy, if $\forall\ x_0 \in X_{\text{opt}} \setminus \tilde{X}_r$, $\forall\ y \notin Y_{\text{opt}}$ and $\forall\ \tilde{x}_r \in \tilde{X}_r$ there is the identity $V_{\text{opt}} = K(x_0, y)$ and the nonstrict inequality $V_{\text{opt}} \leqslant K(\tilde{x}_r, y)$, but $\forall\ \tilde{x}_r \in \tilde{X}_r\ \ \exists\ y \in Y \setminus Y_{\text{opt}}$ that $V_{\text{opt}} < K(\tilde{x}_r,\, y)$, where $\tilde{X}_r \subseteq X_{\text{opt}}$ is the set of all the nonstrictly rational pure strategies of the first player.

Definition 4. In the antagonistic game with the kernel $K(x,\, y)$ an optimal pure strategy $\tilde{y}_r \in \tilde{Y}_r \subset Y_{\text{opt}}$ of the second player is called the nonstrictly rational pure strategy, if $\forall\ y_0 \in Y_{\text{opt}} \setminus \tilde{Y}_r$, $\forall\ x \notin X_{\text{opt}}$ and $\forall\ \tilde{y}_r \in \tilde{Y}_r$ there is the identity $K(x,\, y_0) = V_{\text{opt}}$ and the nonstrict inequality $K(x,\, \tilde{y}_r) \leqslant V_{\text{opt}}$, but $\forall\ \tilde{y}_r \in \tilde{Y}_r$ $\exists\ x \in X \setminus X_{\text{opt}}$ that $K(x,\, \tilde{y}_r) < V_{\text{opt}}$, where $\tilde{Y}_r \subseteq Y_{\text{opt}}$ is the set of all the nonstrictly rational pure strategies of the second player.

If $\tilde{S}_r \neq \varnothing$ and $\left|S_{\text{opt}}\right| > 1$, then a player must select a nonstrictly rational pure strategy $\tilde{s}_r \in \tilde{S}_r$, which would provide some advantage in comparison with another optimal pure strategy $s_{\text{opt}} \in S_{\text{opt}}$. On this ground there may be defined the most advantaged optimal pure strategies from the set $\tilde{S}_r$ or, generally, from the set S_{opt}. Those optimal strategies may be named as absolutely optimal or superoptimal.

Definition 5. In the antagonistic game with the kernel $K(x,\, y)$ by $X_{\text{opt}} \neq \varnothing$, $Y_{\text{opt}} \neq \varnothing$, and $\left|Y \setminus Y_{\text{opt}}\right| \in \mathbb{N} \setminus \{1\}$, an optimal pure strategy $\widehat{x}_{\text{opt}} \in X_{\text{opt}}$ of the first player is called the superoptimal pure strategy, if there are at least two pure strategies $x_{\text{opt}}^{\langle 1 \rangle} \in X_{\text{opt}}$ and $x_{\text{opt}}^{\langle 2 \rangle} \in X_{\text{opt}}$ that

$$\sum_{y \notin Y_{\text{opt}}} K\left(x_{\text{opt}}^{\langle 1 \rangle},\, y\right) \neq \sum_{y \notin Y_{\text{opt}}} K\left(x_{\text{opt}}^{\langle 2 \rangle},\, y\right), \tag{1}$$

and by the Bayes-Laplace criterion [8]

$$\widehat{x}_{\text{opt}} \in \arg\max_{x \in X_{\text{opt}}} \sum_{y \notin Y_{\text{opt}}} K(x,\, y) \tag{2}$$

or by the multiplication criterion [8]

$$\widehat{x}_{\text{opt}} \in \left\{ \arg\max_{x \in X_{\text{opt}}} \prod_{y \notin Y_{\text{opt}}} \left(K(x,\, y) + \left[c - \min_{x \in X} \min_{y \in Y} K(x,\, y) \right] \cdot \operatorname{sign}\left[1 - \operatorname{sign} \min_{x \in X} \min_{y \in Y} K(x,\, y) \right] \right) \right\}, \tag{3}$$

where $c > 0$. The set of all the superoptimal pure strategies of the first player is

$$\widehat{X}_{\text{opt}} = \left\{ \arg\max_{x \in X_{\text{opt}}} \sum_{y \notin Y_{\text{opt}}} K(x,\, y) \right\} \subset \tilde{X}_r \subset X_{\text{opt}} \tag{4}$$

by the Bayes-Laplace criterion, or

$$\widehat{X}_{\text{opt}}(c) = \left\{ \arg\max_{x \in X_{\text{opt}}} \prod_{y \notin Y_{\text{opt}}} \left(K(x,\, y) + \left[c - \min_{x \in X} \min_{y \in Y} K(x,\, y) \right] \cdot \operatorname{sign}\left[1 - \right.\right.\right.$$

$$\left.\left.\left. - \operatorname{sign} \min_{x \in X} \min_{y \in Y} K(x,\, y) \right] \right) \right\} \subset \tilde{X}_r \subset X_{\text{opt}} \tag{5}$$

by the multiplication criterion.

Definition 6. In the antagonistic game with the kernel $K(x,\, y)$ by $X_{\text{opt}} \neq \varnothing$, $Y_{\text{opt}} \neq \varnothing$, and $\left|Y \setminus Y_{\text{opt}}\right| = \infty$, an optimal pure strategy $\widehat{x}_{\text{opt}} \in X_{\text{opt}}$ of the first player is called the superoptimal pure strategy, if there are at least two pure strategies $x_{\text{opt}}^{\langle 1 \rangle} \in X_{\text{opt}}$ and $x_{\text{opt}}^{\langle 2 \rangle} \in X_{\text{opt}}$ that

$$\int_{y \notin Y_{\text{opt}}} K\left(x_{\text{opt}}^{\langle 1 \rangle},\, y\right) dy \neq \int_{y \notin Y_{\text{opt}}} K\left(x_{\text{opt}}^{\langle 2 \rangle},\, y\right) dy, \tag{6}$$

and by the Bayes-Laplace criterion

$$\widehat{x}_{\text{opt}} \in \arg\max_{x \in X_{\text{opt}}} \int_{y \notin Y_{\text{opt}}} K(x,\, y)\, dy\,. \tag{7}$$

The set of all the superoptimal pure strategies of the first player is

$$\widehat{X}_{\text{opt}} = \left\{ \arg\max_{x \in X_{\text{opt}}} \int_{y \notin Y_{\text{opt}}} K(x, y)\, dy \right\} \subset \tilde{X}_r \subset X_{\text{opt}} \tag{8}$$

by the Bayes-Laplace criterion.

Definition 7. In the antagonistic game with the kernel $K(x, y)$ by $X_{\text{opt}} \neq \varnothing$, $Y_{\text{opt}} \neq \varnothing$, and $\left| X \setminus X_{\text{opt}} \right| \in \mathbb{N} \setminus \{1\}$, an optimal pure strategy $\breve{y}_{\text{opt}} \in Y_{\text{opt}}$ of the second player is called the superoptimal pure strategy, if there are at least two pure strategies $y_{\text{opt}}^{\langle 1 \rangle} \in Y_{\text{opt}}$ and $y_{\text{opt}}^{\langle 2 \rangle} \in Y_{\text{opt}}$ that

$$\sum_{x \notin X_{\text{opt}}} K\left(x, y_{\text{opt}}^{\langle 1 \rangle}\right) \neq \sum_{x \notin X_{\text{opt}}} K\left(x, y_{\text{opt}}^{\langle 2 \rangle}\right), \tag{9}$$

and by the Bayes-Laplace criterion

$$\breve{y}_{\text{opt}} \in \arg\min_{y \in Y_{\text{opt}}} \sum_{x \notin X_{\text{opt}}} K(x, y) \tag{10}$$

or by the multiplication criterion

$$\breve{y}_{\text{opt}} \in \left\{ \arg\min_{y \in Y_{\text{opt}}} \prod_{x \notin X_{\text{opt}}} \left(K(x, y) + \left[c - \min_{x \in X} \min_{y \in Y} K(x, y) \right] \cdot \operatorname{sign}\left[1 - \operatorname{sign} \min_{x \in X} \min_{y \in Y} K(x, y) \right] \right) \right\}, \tag{11}$$

where $c > 0$. The set of all the superoptimal pure strategies of the second player is

$$\breve{Y}_{\text{opt}} = \left\{ \arg\min_{y \in Y_{\text{opt}}} \sum_{x \notin X_{\text{opt}}} K(x, y) \right\} \subset \tilde{Y}_r \subset Y_{\text{opt}} \tag{12}$$

by the Bayes-Laplace criterion, or

$$\breve{Y}_{\text{opt}}(c) = \left\{ \arg\min_{y \in Y_{\text{opt}}} \prod_{x \notin X_{\text{opt}}} \left(K(x, y) + \left[c - \min_{x \in X} \min_{y \in Y} K(x, y) \right] \cdot \operatorname{sign}\left[1 - \right.\right.\right.$$
$$\left.\left.\left. - \operatorname{sign} \min_{x \in X} \min_{y \in Y} K(x, y) \right] \right) \right\} \subset \tilde{Y}_r \subset Y_{\text{opt}} \tag{13}$$

by the multiplication criterion.

Definition 8. In the antagonistic game with the kernel $K(x, y)$ by $X_{\text{opt}} \neq \varnothing$, $Y_{\text{opt}} \neq \varnothing$, and $\left| X \setminus X_{\text{opt}} \right| = \infty$, an optimal pure strategy $\breve{y}_{\text{opt}} \in Y_{\text{opt}}$ of the second player is called the superoptimal pure strategy, if there are at least two pure strategies $y_{\text{opt}}^{\langle 1 \rangle} \in Y_{\text{opt}}$ and $y_{\text{opt}}^{\langle 2 \rangle} \in Y_{\text{opt}}$ that

$$\int_{x \notin X_{\text{opt}}} K\left(x, y_{\text{opt}}^{\langle 1 \rangle}\right) dx \neq \int_{x \notin X_{\text{opt}}} K\left(x, y_{\text{opt}}^{\langle 2 \rangle}\right) dx, \tag{14}$$

and by the Bayes-Laplace criterion

$$\breve{y}_{\text{opt}} \in \arg\min_{y \in Y_{\text{opt}}} \int_{x \notin X_{\text{opt}}} K(x, y)\, dx. \tag{15}$$

The set of all the superoptimal pure strategies of the second player is

$$\breve{Y}_{\text{opt}} = \left\{ \arg\min_{y \in Y_{\text{opt}}} \int_{x \notin X_{\text{opt}}} K(x, y)\, dx \right\} \subset \tilde{Y}_r \subset Y_{\text{opt}} \tag{16}$$

by the Bayes-Laplace criterion.

For understanding the last four definitions consider an example. Let the matrix

$$K(x,\,y)=\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -4 & 0 & -2 & -1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -1 & -3 & 0 & -2 & -2 & 0 & 2 & -6 & -9 \end{bmatrix} \tag{17}$$

be the kernel of the antagonistic game, where the row number k corresponds to the pure strategy x_k of the first player $\forall\, k=\overline{1,\,6}$, and the column number l corresponds to the pure strategy y_l of the second player $\forall\, l=\overline{1,\,10}$. Consequently, here the pure strategies sets of the players are $X=\{x_k\}_{k=1}^{6}$ and $Y=\{y_l\}_{l=1}^{10}$. Clearly, that this game is solved in pure strategies with the game value $V_{\text{opt}}=0$, and the set

$$X_{\text{opt}}=\{x_1,\,x_2,\,x_4,\,x_5\}, \tag{18}$$

the set

$$Y_{\text{opt}}=\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}. \tag{19}$$

By the four first definitions in the exampled game there are the following primarily advantaged subsets of the sets (18) and (19). The set of the nonstrictly rational pure strategies of the first player is

$$\tilde{X}_r=\{x_1,\,x_4,\,x_5\}=\{X_{\text{opt}}\setminus\{x_2\}\}\subset X_{\text{opt}}=\{x_1,\,x_2,\,x_4,\,x_5\}, \tag{20}$$

and the set of the nonstrictly rational pure strategies of the second player is

$$\tilde{Y}_r=\{y_6,\,y_9,\,y_{10}\}=\{Y_{\text{opt}}\setminus\{y_4,\,y_7\}\}\subset Y_{\text{opt}}=\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}. \tag{21}$$

Deeper, the set of the strictly rational pure strategies of the first player is

$$X_r=\{x_4\}=\{\tilde{X}_r\setminus\{x_1,\,x_5\}\}\subset\tilde{X}_r=\{x_1,\,x_4,\,x_5\}=\{X_{\text{opt}}\setminus\{x_2\}\}\subset X_{\text{opt}}=\{x_1,\,x_2,\,x_4,\,x_5\}, \tag{22}$$

and the set of the strictly rational pure strategies of the second player is

$$Y_r=\{y_6,\,y_9\}=\{\tilde{Y}_r\setminus\{y_{10}\}\}\subset\tilde{Y}_r=\{y_6,\,y_9,\,y_{10}\}=\{Y_{\text{opt}}\setminus\{y_4,\,y_7\}\}\subset Y_{\text{opt}}=\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}. \tag{23}$$

The set of the superoptimal pure strategies of the first player is

$$\widehat{X}_{\text{opt}}=\left\{\arg\max_{x\in X_{\text{opt}}}\sum_{y\notin Y_{\text{opt}}}K(x,\,y)\right\}=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\sum_{y\notin\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}}K(x,\,y)\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\{(1+2+0+0+5),\,(0+0+0+0+0),\,(2+1+1+1+1),\,(0+0+1+0+0)\}\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\{8,\,0,\,6,\,1\}\right\}=\{x_1\}=\{\tilde{X}_r\setminus\{x_4,\,x_5\}\} \tag{24}$$

by the Bayes-Laplace criterion with the formula (4). By the multiplication criterion this set is

$$\widehat{X}_{\text{opt}}(1)=\left\{\arg\max_{x\in X_{\text{opt}}}\prod_{y\notin Y_{\text{opt}}}\left(K(x,\,y)+\left[1-\min_{x\in X}\min_{y\in Y}K(x,\,y)\right]\cdot\operatorname{sign}\left[1-\operatorname{sign}\min_{x\in X}\min_{y\in Y}K(x,\,y)\right]\right)\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\prod_{y\notin\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}}\left(K(x,\,y)+[1-(-9)]\cdot\operatorname{sign}[1-\operatorname{sign}(-9)]\right)\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\prod_{y\notin\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}}\left[K(x,\,y)+10\cdot\operatorname{sign}(2)\right]\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\prod_{y\notin\{y_4,\,y_6,\,y_7,\,y_9,\,y_{10}\}}\left[K(x,\,y)+10\right]\right\}=$$

$$=\left\{\arg\max_{x\in\{x_1,\,x_2,\,x_4,\,x_5\}}\{(11\cdot12\cdot10\cdot10\cdot15),\,(10\cdot10\cdot10\cdot10\cdot10),\,(12\cdot11\cdot11\cdot11\cdot11),\,(10\cdot10\cdot11\cdot10\cdot10)\}\right\}=$$

$$= \left\{ \arg \max_{x \in \{x_1, x_2, x_4, x_5\}} \{198000, 100000, 175692, 110000\} \right\} = \{x_1\} = \left\{ \tilde{X}_r \setminus \{x_4, x_5\} \right\} \qquad (25)$$

with the formula (5) for $c = 1$. It is situational that the needed set, been found by two criterions, is the same. But generally, there exist such antagonistic games, where $\exists\, c > 0$ there is the statement

$$\widehat{X}_{\text{opt}} \cap \widehat{X}_{\text{opt}}(c) = \varnothing . \qquad (26)$$

The set of the superoptimal pure strategies of the second player is

$$\breve{Y}_{\text{opt}} = \left\{ \arg \min_{y \in Y_{\text{opt}}} \sum_{x \notin X_{\text{opt}}} K(x, y) \right\} = \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \sum_{x \notin \{x_1, x_2, x_4, x_5\}} K(x, y) \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \{(0+0), (-4-2), (0+0), (-1-6), (0-9)\} \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \{0, -6, 0, -7, -9\} \right\} = \{y_{10}\} = \left\{ \tilde{Y}_r \setminus \{y_6, y_9\} \right\} \qquad (27)$$

by the Bayes-Laplace criterion with the formula (12). By the multiplication criterion this set is

$$\breve{Y}_{\text{opt}}(1) = \left\{ \arg \min_{y \in Y_{\text{opt}}} \prod_{x \notin X_{\text{opt}}} \left(K(x, y) + \left[1 - \min_{x \in X} \min_{y \in Y} K(x, y) \right] \cdot \text{sign} \left[1 - \text{sign} \min_{x \in X} \min_{y \in Y} K(x, y) \right] \right) \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \prod_{x \notin \{x_1, x_2, x_4, x_5\}} \left(K(x, y) + [1 - (-9)] \cdot \text{sign}[1 - \text{sign}(-9)] \right) \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \prod_{x \notin \{x_1, x_2, x_4, x_5\}} [K(x, y) + 10 \cdot \text{sign}(2)] \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \prod_{x \notin \{x_1, x_2, x_4, x_5\}} [K(x, y) + 10] \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \{(10 \cdot 10), (6 \cdot 8), (10 \cdot 10), (9 \cdot 4), (10 \cdot 1)\} \right\} =$$

$$= \left\{ \arg \min_{y \in \{y_4, y_6, y_7, y_9, y_{10}\}} \{100, 48, 100, 36, 10\} \right\} = \{y_{10}\} = \left\{ \tilde{Y}_r \setminus \{y_6, y_9\} \right\} \qquad (28)$$

with the formula (13) for $c = 1$. And once again the needed set, been found by two criterions, is the same. But generally, there exist such antagonistic games, where $\exists\, c > 0$ there is the statement

$$\breve{Y}_{\text{opt}} \cap \breve{Y}_{\text{opt}}(c) = \varnothing . \qquad (29)$$

Hence, if in the exampled game the first player applies the superoptimal pure strategy x_1 then in average it obtains the greatest advantage when the second player swerves from applying the set Y_{opt}. The same could be said about the situation when the first player swerves from applying the set X_{opt}. In that case if the second player applies the superoptimal pure strategy y_{10} then in average it obtains the greatest advantage. Nevertheless, applying the sets $\widehat{X}_{\text{opt}}$ and $\breve{Y}_{\text{opt}}$ ensures both players in obtaining the mathematically expected greatest advantage, rather than the greatest advantage in the single play. Say, if the second player swerves from applying the set Y_{opt} and selects the pure strategy y_3 then the first player payoff is V_{opt}, though if it selected a non-superoptimal pure strategy x_4 or x_5, it would obtain the payoff $V_{\text{opt}} + 1$. Besides, holding at the strictly rational pure strategy x_4 would ensure the first player in obtaining the payoff, which is greater than V_{opt}, for every time when the second player swerves from applying the set Y_{opt}. So, the stated above superoptimality concept for pure strategies is acceptable for the cases, when the applied superoptimal pure strategy is a strictly rational pure strategy or there is the sufficiently great number of the game plays, that will allow to obtain the mathematically expected greatest advantage.

Henceforward will consider antagonistic games, which generally are solved in mixed strategies. May

$p_{\text{opt}}(x)$ be an optimal mixed strategy of the first player, and $q_{\text{opt}}(y)$ be an optimal mixed strategy of the second player, which satisfy the double inequality

$$\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p(x) q_{\text{opt}}(y)\, dxdy \leqslant V_{\text{opt}} =$$

$$= \int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}(x) q_{\text{opt}}(y)\, dxdy \leqslant \int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}(x) q(y)\, dxdy \qquad (30)$$

by the conditions

$$p(x) \in \left\{ p(x) : p(x) \geqslant 0 \ \forall\ x \in X,\ \int\limits_{x \in X} p(x)\, dx = 1 \right\} = \mathscr{X}, \qquad (31)$$

$$q(y) \in \left\{ q(y) : q(y) \geqslant 0 \ \forall\ y \in Y,\ \int\limits_{y \in Y} q(y)\, dy = 1 \right\} = \mathscr{Y}, \qquad (32)$$

where $p_{\text{opt}}(x) \in \mathscr{X}_{\text{opt}} \subset \mathscr{X}$ and $q_{\text{opt}}(y) \in \mathscr{Y}_{\text{opt}} \subset \mathscr{Y}$. Subsequently, $\mathscr{X}$ is the set of all the mixed strategies of the first player, $\mathscr{Y}$ is the set of all the mixed strategies of the second player, $\mathscr{X}_{\text{opt}}$ is the set of all the optimal mixed strategies of the first player, $\mathscr{Y}_{\text{opt}}$ is the set of all the optimal mixed strategies of the second player. Of course, a mixed strategy (31) of the first player and a mixed strategy (32) of the second player may be implied for matrix games or games with the enumerable set of the pure strategies with the nonzero optimal probabilities. On this ground there are the following definitions for the superoptimality concept in the mixed strategies.

Definition 9. In the antagonistic game with the kernel $K(x, y)$ an optimal mixed strategy $\widehat{p}_{\text{opt}}(x) \in \mathscr{X}_{\text{opt}}$ of the first player is called the superoptimal mixed strategy, if there are at least two mixed strategies $p_{\text{opt}}^{\langle 1 \rangle}(x) \in \mathscr{X}_{\text{opt}}$ and $p_{\text{opt}}^{\langle 2 \rangle}(x) \in \mathscr{X}_{\text{opt}}$ that

$$\int\limits_{q(y) \notin \mathscr{Y}_{\text{opt}}} \left(\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}^{\langle 1 \rangle}(x) q(y)\, dxdy \right) d\left[q(y) \right] \neq$$

$$\neq \int\limits_{q(y) \notin \mathscr{Y}_{\text{opt}}} \left(\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}^{\langle 2 \rangle}(x) q(y)\, dxdy \right) d\left[q(y) \right], \qquad (33)$$

and by the Bayes-Laplace criterion

$$\widehat{p}_{\text{opt}}(x) \in \arg \max_{p_{\text{opt}}(x) \in \mathscr{X}_{\text{opt}}} \int\limits_{q(y) \notin \mathscr{Y}_{\text{opt}}} \left(\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}(x) q(y)\, dxdy \right) d\left[q(y) \right]. \qquad (34)$$

The set of all the superoptimal mixed strategies of the first player is

$$\widehat{\mathscr{X}}_{\text{opt}} = \left\{ \arg \max_{p_{\text{opt}}(x) \in \mathscr{X}_{\text{opt}}} \int\limits_{q(y) \notin \mathscr{Y}_{\text{opt}}} \left(\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p_{\text{opt}}(x) q(y)\, dxdy \right) d\left[q(y) \right] \right\} \subset \mathscr{X}_{\text{opt}} \qquad (35)$$

by the Bayes-Laplace criterion. The exterior integrals in the formulas (33)-(35) are the generalized Riemann integrals [9-12] of the variable $q(y)$ over the subset of the set $\mathscr{Y}$, where the function $q(y) \notin \mathscr{Y}_{\text{opt}}$.

Definition 10. In the antagonistic game with the kernel $K(x, y)$ an optimal mixed strategy $\breve{q}_{\text{opt}}(y) \in \mathscr{Y}_{\text{opt}}$ of the second player is called the superoptimal mixed strategy, if there are at least two mixed strategies $q_{\text{opt}}^{\langle 1 \rangle}(y) \in \mathscr{Y}_{\text{opt}}$ and $q_{\text{opt}}^{\langle 2 \rangle}(y) \in \mathscr{Y}_{\text{opt}}$ that

$$\int\limits_{p(x) \notin \mathscr{X}_{\text{opt}}} \left(\int\limits_{x \in X} \int\limits_{y \in Y} K(x, y)\, p(x) q_{\text{opt}}^{\langle 1 \rangle}(y)\, dxdy \right) d\left[p(x) \right] \neq$$

$$\neq \int\limits_{p(x)\notin \mathscr{X}_{\mathrm{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x,\, y)\, p(x)\, q_{\mathrm{opt}}^{\langle 2\rangle}(y)\, dxdy \right) d\left[p(x) \right], \tag{36}$$

and by the Bayes-Laplace criterion

$$\breve{q}_{\mathrm{opt}}(y) \in \arg \min_{q_{\mathrm{opt}}(y)\in \mathscr{Y}_{\mathrm{opt}}} \int\limits_{p(x)\notin \mathscr{X}_{\mathrm{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x,\, y)\, p(x)\, q_{\mathrm{opt}}(y)\, dxdy \right) d\left[p(x) \right]. \tag{37}$$

The set of all the superoptimal mixed strategies of the second player is

$$\breve{\mathscr{Y}}_{\mathrm{opt}} = \left\{ \arg \min_{q_{\mathrm{opt}}(y)\in \mathscr{Y}_{\mathrm{opt}}} \int\limits_{p(x)\notin \mathscr{X}_{\mathrm{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x,\, y)\, p(x)\, q_{\mathrm{opt}}(y)\, dxdy \right) d\left[p(x) \right] \right\} \subset \mathscr{Y}_{\mathrm{opt}} \tag{38}$$

by the Bayes-Laplace criterion. The exterior integrals in the formulas (36)-(38) are the generalized Riemann integrals of the variable $p(x)$ over the subset of the set $\mathscr{X}$, where the function $p(x) \notin \mathscr{X}_{\mathrm{opt}}$.

For comprehending the last two dual definitions consider an example on the nonstrictly convex game with the kernel [13]

$$K(x,\, y) = ax^2 + bxy + cy + k\,, \tag{39}$$

that is defined on the unit square

$$S_K = X \times Y = [0;\, 1] \times [0;\, 1] \tag{40}$$

with the nonzero coefficients a, b, c and the constant $k \in \mathbb{R}$, where $x \in X = [0;\, 1]$ and $y \in Y = [0;\, 1]$ are the pure strategies of the first player and second player correspondingly, and $\forall\ x \in X$, $\forall\ y \in Y$ there is the nonstrict inequality $\dfrac{\partial^2 K(x,\, y)}{\partial y^2} \geqslant 0$. In the case when $a > 0$, $b < 0$, $c < 0$, $a + b = 0$ [13, p. 187], the maximum of the kernel (39) by the variable x on the unit segment $X = [0;\, 1]$ is [13]

$$\max_{x\in X} K(x,\, y) = \max_{x\in[0;\,1]} K(x,\, y) = \max_{x\in[0;\,1]} \left(ax^2 + bxy + cy + k \right) = \max_{x\in[0;\,1]} \left(ax^2 - axy + cy + k \right) =$$

$$= \max\left\{ K(0,\, y),\, K(1,\, y) \right\} = \max\left\{ cy + k,\, a - ay + cy + k \right\} = a - ay + cy + k = K(1,\, y). \tag{41}$$

The result in the statement (41) is pretty clear, as on the unit segment $X = [0;\, 1]$ the maximum of the parabola (39) as the function of the variable x may be reached either in the point $x = 0$ or $x = 1$, and inasmuch as $a - ay \geqslant 0\ \ \forall\ y \in [0;\, 1]$, then this maximum is reached in the point $x = 1$. Subsequently, the minimum of the line (41) on the unit segment $Y = [0;\, 1]$ is [13]

$$\min_{y\in Y} \max_{x\in X} K(x,\, y) = \min_{y\in[0;\,1]} \max_{x\in[0;\,1]} K(x,\, y) = \min_{y\in[0;\,1]} K(1,\, y) = \min_{y\in[0;\,1]} \left(a - ay + cy + k \right) =$$

$$= \min\left\{ K(1,\, 0),\, K(1,\, 1) \right\} = \min\left\{ a + k,\, a - a + c + k \right\} = \min\left\{ a + k,\, c + k \right\} = c + k = K(1,\, 1) = V_{\mathrm{opt}}. \tag{42}$$

The minimum (42) is reached in the point $y_{\mathrm{opt}} = 1$, that is by the definition

$$Y_{\mathrm{opt}} = \arg \min_{y\in Y} \max_{x\in X} K(x,\, y) = \arg \min_{y\in[0;\,1]} K(1,\, y) = \arg \min_{y\in[0;\,1]} \left(a - ay + cy + k \right) = \{1\} = \left\{ y_{\mathrm{opt}} \right\}. \tag{43}$$

The solutions of the corresponding equation [13]

$$V_{\mathrm{opt}} = c + k = K(1,\, 1) = ax^2 + bx + c + k = ax^2 - ax + c + k = K(x,\, 1) = K\left(x,\, y_{\mathrm{opt}}\right) \tag{44}$$

are $x_1 = 0$ and $x_2 = 1$. However, here are the negative values [13]

$$\left. \frac{dK(x_1,\, y)}{dy} \right|_{y=y_{\mathrm{opt}}} = \left. \frac{d\left(ax_1^2 + bx_1 y + cy + k \right)}{dy} \right|_{y=y_{\mathrm{opt}}} = bx_1 + c = r_1 = -ax_1 + c = c\,, \tag{45}$$

$$\left. \frac{dK(x_2,\, y)}{dy} \right|_{y=y_{\mathrm{opt}}} = \left. \frac{d\left(ax_2^2 + bx_2 y + cy + k \right)}{dy} \right|_{y=y_{\mathrm{opt}}} = bx_2 + c = r_2 = -ax_2 + c = c - a\,, \tag{46}$$

and, properly, the equation [13]

$$P(x_1)r_1 + P(x_2)r_2 = P(x_1)r_1 + \left[1 - P(x_1)\right]r_2 = 0 \tag{47}$$

for the probabilities $P(x_1)$ and $P(x_2)$ of selecting the pure strategies $x_1 = 0$ and $x_2 = 1$, where $P(x_1) + P(x_2) = 1$, has no sense. Then, including the set $X_{\text{opt}} = \{x_1, x_2\} = \{0, 1\}$, the optimal mixed strategy of the first player is

$$p_{\text{opt}}(x) \in \mathscr{X}_{\text{opt}} = \left\{ p_{\text{opt}}(x) : p_{\text{opt}}(x) \geqslant 0 \ \forall\ x \in [0; 1],\ p_{\text{opt}}(x) = 0 \ \forall\ x \in (0; 1),\ \int_0^1 p_{\text{opt}}(x)dx = 1 \right\}. \tag{48}$$

Actually, the set (48) may be stated in the optimal probabilities of selecting the pure strategies $x_1 = 0$ and $x_2 = 1$, that is

$$P_{\text{opt}}(0) \in [0; 1],\ P_{\text{opt}}(1) \in [0; 1],\ P_{\text{opt}}(0) + P_{\text{opt}}(1) = 1. \tag{49}$$

Then by Definition 9 firstly will state for $u \in \{1, 2\}$ the expected payoff of the first player:

$$\int_{x\in X}\int_{y\in Y} K(x, y)\, p_{\text{opt}}^{\langle u\rangle}(x) q(y)\, dxdy = P_{\text{opt}}^{\langle u\rangle}(0) \int_{y\in[0; 1]} K(0, y) q(y) dy + P_{\text{opt}}^{\langle u\rangle}(1) \int_{y\in[0; 1]} K(1, y) q(y) dy =$$

$$= P_{\text{opt}}^{\langle u\rangle}(0) \int_0^1 K(0, y) q(y) dy + \left[1 - P_{\text{opt}}^{\langle u\rangle}(0)\right] \int_0^1 K(1, y) q(y) dy =$$

$$= \int_0^1 K(1, y) q(y) dy + P_{\text{opt}}^{\langle u\rangle}(0) \left(\int_0^1 K(0, y) q(y) dy - \int_0^1 K(1, y) q(y) dy \right) =$$

$$= \int_0^1 \left[K(1, y) + K(0, y) P_{\text{opt}}^{\langle u\rangle}(0) - K(1, y) P_{\text{opt}}^{\langle u\rangle}(0) \right] q(y) dy =$$

$$= \int_0^1 \left[(a - ay + cy + k) + (cy + k) \cdot P_{\text{opt}}^{\langle u\rangle}(0) - (a - ay + cy + k) \cdot P_{\text{opt}}^{\langle u\rangle}(0) \right] q(y) dy =$$

$$= \int_0^1 \left[a - ay + cy + k + cyP_{\text{opt}}^{\langle u\rangle}(0) + kP_{\text{opt}}^{\langle u\rangle}(0) - aP_{\text{opt}}^{\langle u\rangle}(0) + ayP_{\text{opt}}^{\langle u\rangle}(0) - cyP_{\text{opt}}^{\langle u\rangle}(0) - kP_{\text{opt}}^{\langle u\rangle}(0) \right] q(y) dy =$$

$$= \int_0^1 \left[a - ay + cy + k - aP_{\text{opt}}^{\langle u\rangle}(0) + ayP_{\text{opt}}^{\langle u\rangle}(0) \right] q(y) dy =$$

$$= \int_0^1 \left[y\left(aP_{\text{opt}}^{\langle u\rangle}(0) + c - a \right) - aP_{\text{opt}}^{\langle u\rangle}(0) + a + k \right] q(y) dy. \tag{50}$$

The integration in the formula (33) over the set of the mixed strategies of the second player (32), where each element of this set $q(y) \notin \mathscr{Y}_{\text{opt}}$, is represented as the generalized Riemann integral with the subintegral function (50). If some optimal mixed strategy of the first player $p_{\text{opt}}^{\langle 1\rangle}(x)$ is $P_{\text{opt}}^{\langle 1\rangle}(0) = 0$ then in the left side of the inequality (33) have

$$\int_{q(y)\notin \mathscr{Y}_{\text{opt}}} \left(\int_{x\in X}\int_{y\in Y} K(x, y)\, p_{\text{opt}}^{\langle 1\rangle}(x) q(y)\, dxdy \right) d\left[q(y)\right] =$$

$$= \int_{q(y)\notin \mathscr{Y}_{\text{opt}}} \left(\int_0^1 \left[y(c - a) + a + k \right] q(y) dy \right) d\left[q(y)\right]. \tag{51}$$

And if some optimal mixed strategy of the first player $p_{\text{opt}}^{\langle 2\rangle}(x)$ is $P_{\text{opt}}^{\langle 2\rangle}(0) = 1$ then in the right side of the inequality (33) have

$$\int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x, y)\, p_{\text{opt}}^{\langle 2\rangle}(x) q(y)\, dx dy \right) d\left[q(y)\right] = \int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_{0}^{1} [cy+k] q(y)\, dy \right) d\left[q(y)\right]. \qquad (52)$$

Inasmuch as $\forall\ y \in [0;\, 1]$ within the subintegral functions of (51) and (52) there is the statement

$$y(c-a)+a+k-(cy+k)=a-ay\geqslant 0, \qquad (53)$$

which turns into the equality by $y=1$ and every mixed strategy of the second player $q(y)\notin\mathscr{Y}_{\text{opt}}$ belongs to the set

$$\mathscr{Y}\setminus\mathscr{Y}_{\text{opt}} = \left\{ q(y): q(y)\geqslant 0\ \forall\ y\in[0;\,1],\ \int\limits_0^1 q(y)\, dy = 1 \right\} \setminus$$

$$\setminus \left\{ q(y): q(y)=0\ \forall\ y\in[0;\,1),\ q(1)>0,\ \int\limits_0^1 q(y)\, dy = 1 \right\}, \qquad (54)$$

then it is clear that the generalized Riemann integrals (51) and (52) have different values:

$$\int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x, y)\, p_{\text{opt}}^{\langle 1\rangle}(x) q(y)\, dx dy \right) d\left[q(y)\right] =$$

$$= \int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_0^1 \left[y(c-a)+a+k\right] q(y)\, dy \right) d\left[q(y)\right] \neq \int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_0^1 [cy+k] q(y)\, dy \right) d\left[q(y)\right] =$$

$$= \int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x, y)\, p_{\text{opt}}^{\langle 2\rangle}(x) q(y)\, dx dy \right) d\left[q(y)\right]. \qquad (55)$$

Accordingly, the inequality (33) in Definition 9 is true, and there is the nonempty set of the superoptimal mixed strategies of the first player in the being exampled game. Here, actually, for determining the set (35) it is sufficient to determine the maximum of the statement

$$\varphi\left[P_{\text{opt}}(0)\right] = y\left(aP_{\text{opt}}^{\langle u\rangle}(0)+c-a\right) - aP_{\text{opt}}^{\langle u\rangle}(0)+a+k = (ay-a)P_{\text{opt}}(0)+a+(c-a)y+k \qquad (56)$$

within the subintegral function (50) by the variable $P_{\text{opt}}(0)$ on the segment $[0;\, 1]$. With the inequality (53) it is obvious, that the line (56) on the segment $[0;\, 1]$ has its maximum in the point $P_{\text{opt}}(0)=0$. Subsequently, this point designates the set

$$\widehat{\mathscr{X}}_{\text{opt}} = \left\{ \arg \max_{P_{\text{opt}}(x)\subset\mathscr{X}_{\text{opt}}} \int\limits_{q(y)\notin\mathscr{Y}_{\text{opt}}} \left(\int\limits_{x\in X} \int\limits_{y\in Y} K(x, y)\, p_{\text{opt}}(x) q(y)\, dx dy \right) d\left[q(y)\right] \right\} =$$

$$= \left\{ p(x): p(x)=0\ \forall\ x\in[0;\,1),\ p(1)>0,\ \int\limits_0^1 p(x)\, dx = 1 \right\} \subset \mathscr{X}_{\text{opt}} \qquad (57)$$

of the single superoptimal mixed strategy of the first player. Applying the single element of this set, that is $P_{\text{opt}}(1)=1$ or, more clearly, applying the pure strategy $x=x_{\text{opt}}=1$, the first player gets the maximized advantage as soon as the second player swerves from applying the single pure strategy $y_{\text{opt}}=1$. For instance, if the second player selects $y=0.9$ owing to some unknown motives, then by applying the set (57) the first player gets the payoff, equal to the value

$$K(1,\, 0.9) = a - a\cdot 0.9 + c\cdot 0.9 + k = 0.1a + 0.9c + k\,. \qquad (58)$$

In the same situation, if the first player had selected the strategy, that is not the superoptimal mixed strategy from the set (57), say, if $P_{\text{opt}}(0)=0.3$ then the expected payoff would have been equal to the value

$$K(0,\, 0.9)\cdot P_{\text{opt}}(0) + K(1,\, 0.9)\cdot P_{\text{opt}}(1) = K(0,\, 0.9)\cdot P_{\text{opt}}(0) + K(1,\, 0.9)\cdot\left[1-P_{\text{opt}}(0)\right] =$$

$$= 0.3K(0,\, 0.9) + 0.7K(1,\, 0.9) = 0.3(c\cdot 0.9+k) + 0.7(a - a\cdot 0.9 + c\cdot 0.9 + k) =$$

$$= 0.27c + 0.3k + 0.07a + 0.63c + 0.7k = 0.07a + 0.9c + k\,. \qquad (59)$$

And here apparently, that the value (58) is greater than the value (59) for the positive number $0.03a$.

As for the second player, then in this exampled game $\breve{Y}_{\mathrm{opt}} = \varnothing$ or, more generally, $\breve{\mathscr{Y}}_{\mathrm{opt}} = \varnothing$. It flows outright from that the inequality (14) or more general inequality (36) cannot be true.

Conclusion and further investigation prospect. There exist such antagonistic games, where may be determined and applied as the most advantaged optimal pure strategies, as well as the most advantaged optimal mixed strategies of a player. Application of these advantaged subsets of the optimal mixed strategies set gives the most great potential profit. For the player, which is applying the most advantaged optimal mixed strategies, named the superoptimal mixed strategies, this potential is implemented every time, when the other player swerves from applying its optimal mixed strategies. The defined sets of the superoptimal mixed strategies, which are found by (35) and (38), may be considered as the generalization of the pure strategies superoptimality, stated in the paper [4]. Therefore, the players are recommended instead of their optimal mixed strategies sets $\mathscr{X}_{\mathrm{opt}}$ and $\mathscr{Y}_{\mathrm{opt}}$ to apply their superoptimal mixed strategies sets $\widehat{\mathscr{X}}_{\mathrm{opt}}$ and $\breve{\mathscr{Y}}_{\mathrm{opt}}$. The investigated above superoptimality is based on the Bayes-Laplace criterion. However, there remain some questions of the analytical calculation of the generalized Riemann integral [14], been figured in the formulas (33)-(38). Though for some convex games, like the exampled game with the kernel (39), it is easy to evaluate the emergent generalized Riemann integrals, the further investigation prospect should be viewed at peculiarities of that integration.

РЕЗЮМЕ

Означено множину оптимальних змішаних стратегій з найбільшою перевагою, названих супероптимальними змішаними стратегіями, для їх застосування та отримання потенційної корисності у відповідних антагоністичних іграх. Викладений принцип супероптимальності заснований на критерії Байєса-Лапласа.

Ключові слова: антагоністична гра, оптимальна змішана стратегія, критерій Байєса-Лапласа.

РЕЗЮМЕ

Определено множество оптимальных смешанных стратегий с наибольшим преимуществом, названных супероптимальными смешанными стратегиями, для их применения и получения потенциальной полезности в подходящих антагонистических играх. Изложенный принцип супероптимальности основан на критерии Байеса-Лапласа.

Ключевые слова: антагонистическая игра, оптимальная смешанная стратегия, критерий Байеса-Лапласа.

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Received May 7, 2010